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LRS Bianchi Type II Perfect Fluid Cosmological Models in Normal Gauge for Lyra's Manifold

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Abstract The present study deals with spatially homogeneous and locally rotationally symmetric (LRS) Bianchi type II cosmological models of perfect fluid distribution of matter for the field equations in normal gauge for Lyra's manifold where gauge function β is taken as time dependent. To get the deterministic models of the universe, we assume that the expansion (θ) in the model is proportional to the shear (σ). This leads to condition $R = mS^n$, where R and S are metric potentials, m and n are constants. We have obtained two types of models of the universe for two different values of n. It has been found that the displacement vector β behaves like cosmological term Λ in the normal gauge treatment and the solutions are consistent with recent observations. Some physical and geometric behavior of these models are also discussed.

Keywords Cosmology · LRS Bianchi type II models · Lyra's manifold

1 Introduction

In Einstein's general theory, the curvature of a space-time is influenced by matter, and provides the geometrical description of matter. Einstein (1917) succeeded in geometrizing gravitation by expressing gravitational potential in terms of metric tensor. Weyl, in 1918, was inspired by it and he was the first to unify gravitation and electromagnetism in a single space-time geometry. He showed how can one introduce a vector field in the Riemannian

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space-time with an intrinsic geometrical significance. But this theory was not accepted as it was based on non-integrability of length transfer. Lyra [1] introduced a gauge function, i.e., a displacement vector in Riemannian space-time which removes the non-integrability condition of a vector under parallel transport. In this way Riemannian geometry was given a new modification by him and the modified geometry was named as Lyra's manifold.

Sen [2] and Sen and Dunn [3] have proposed a new scalar-tensor theory of gravitation and constructed the field equations analogous to the Einstein's field equations, based on Lyra's manifold which in normal gauge may be written in the form

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -8\pi T_{ij}, \tag{1}$$

where $\phi_i = (0, 0, 0, \beta(t))$ is the displacement vector and other symbols have their usual meanings as in Reimannian geometry. Though the displacement vector has no clear and unambiguous interpretation some efforts have been made to treat the constant displacement vector as the analog of cosmological constant that enters equations in an asymmetric way.

Halford [4] has pointed out that the constant vector displacement field ϕ_i in Lyra's manifold plays the role of cosmological constant Λ in the normal general relativistic treatment. It is shown by Halford [5] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects, within observational limits as the Einstein's theory. The Sen [2] theory and its more generalizations (Sen and Dun [3]; Sen and Vanstone [6]) have received considerable attention in cosmological context. Several investigators [6–27] have studied cosmological models based on Lyra's manifold in different contexts. Soleng [7] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector ϕ will either include a creation field and be equal to Hoyle's creation field cosmology [28–30] or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

Recently, Pradhan et al. [31–37], Casama et al. [38], Bali and Chandnani [39, 40], Kumar and Singh [41], Ram, Zeyauddin and Singh [42], Singh [43] and Rao, Vinutha and Santhi [44] have studied cosmological models based on Lyra's geometry in various contexts. With these motivations, in this paper, we have obtained LRS Bianchi type II cosmological models of perfect fluid distribution of matter for the field equations in normal gauge for Lyra's manifold where gauge function β is taken as time dependent. This paper is organized as follows. In Sect. 1 the motivation for the present work is discussed. The metric and the field equations are presented in Sect. 2. Section 3 deals with general solution. Sections 4 and 5 describe two cases of the solutions of field equations to obtain first and second models with their physical and geometric aspects respectively. The effect of entropy on displacement vector β is given in Sect. 6. Finally, in Sect. 7 discussion and concluding remarks are given.

2 The Metric and Basic Equations

We consider LRS Bianchi type metric in the form

$$ds^2 = \eta_{ab}\theta^a\theta^b, \tag{2}$$

where

$$\theta^1 = Rdx, \quad \theta^2 = S(dy - xdz), \quad \theta^3 = Rdz, \quad \theta^4 = dt. \tag{3}$$

Thus, the metric (2) leads to

$$ds^2 = -dt^2 + R^2dx^2 + S^2(dy - xdz)^2 + R^2dz^2, \tag{4}$$

where R and S are functions of cosmic time t only.

The spatial volume of this model is given by

$$V^3 = R^2S. \tag{5}$$

Here, we also define $V = (R^2S)^{\frac{1}{3}}$ as the average scale factor so that Hubble's parameter is defined by

$$H = \frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right), \tag{6}$$

where an over dot denotes differentiation with respect to the cosmic time t.

The energy momentum tensor for bulk viscous fluid distribution is taken as

$$T_i^j = (\rho + p)v_i v^j + pg_i^j, \tag{7}$$

where ρ, and p are energy density and isotropic pressure respectively and $v^i = \frac{dx^i}{ds}$ is the four-velocity satisfying the condition

$$g_{ij}v^i v^j = -1. \tag{8}$$

We assume that coordinates to be co-moving so that $v^1 = 0 = v^2 = v^3$, $v^4 = 1$. The displacement field vector ϕ_i in the field equation (1) is defined by

$$\phi_i = (0, 0, 0, \beta(t)). \tag{9}$$

For the line element (2), the field equation (1) with (7) and (9) lead to the following system of three independent equations

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{3}{4}\frac{S^2}{R^4} + \frac{3}{4}\beta^2 = -8\pi p, \tag{10}$$

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} + \frac{3}{4}\beta^2 = -8\pi p, \tag{11}$$

$$\frac{2\dot{R}\dot{S}}{RS} + \frac{\dot{R}^2}{R^2} - \frac{1}{4}\frac{S^2}{R^4} - \frac{3}{4}\beta^2 = 8\pi\rho. \tag{12}$$

The energy conservation equation $T^i_{i;j} = 0$ leads to

$$\dot{\rho} + (\rho + p)\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = 0, \tag{13}$$

and conservation of R.H.S. of (1) leads to

$$\left(R_i^j - \frac{1}{2}g_i^j R\right)_{;j} + \frac{3}{2}(\phi_i\phi^j)_{;j} - \frac{3}{4}(g_i^j\phi_k\phi^k)_{;j} = 0. \tag{14}$$

Equation (14) reduces to

$$\frac{3}{2}\phi_i\left[\frac{\partial\phi^j}{\partial x^j}+\phi^l\Gamma^j_{lj}\right]+\frac{3}{2}\phi^j\left[\frac{\partial\phi_i}{\partial x^j}-\phi_l\Gamma^l_{ij}\right]\frac{3}{4}g^j_i\phi_k\left[\frac{\partial\phi^k}{\partial x^j}+\phi^l\Gamma^k_{lj}\right]$$
$$-\frac{3}{4}g^j_i\phi^k\left[\frac{\partial\phi_k}{\partial x^j}-\phi_l\Gamma^l_{kj}\right]=0. \tag{15}$$

Equation (15) is identically satisfied for $i=1,2,3$. For $i=4$, Equation (15) reduces to

$$\frac{3}{2}\beta\left[\frac{\partial(g^{44}\phi_4)}{\partial x^4}+\phi^4\Gamma^4_{44}\right]+\frac{3}{2}g^{44}\phi_4\left[\frac{\partial\phi_4}{\partial t}-\phi_4\Gamma^4_{44}\right]-\frac{3}{4}g^4_4\phi_4\left[\frac{\partial\phi^4}{\partial x^4}+\phi^4\Gamma^4_{44}\right]$$
$$-\frac{3}{4}g^4_4g^{44}\phi^4\left[\frac{\partial\phi_4}{\partial t}-\phi^4\Gamma^4_{44}\right]=0 \tag{16}$$

which leads to

$$\frac{3}{2}\beta\dot{\beta}+\frac{3}{2}\beta^2\left(\frac{2\dot{R}}{R}+\frac{\dot{S}}{S}\right)=0. \tag{17}$$

Thus (13) combined with (17) is the resulting equation when energy conservation equation is satisfied in the given system. It is important to mention here that the conservation equation in Lyra's manifold is not satisfied as in general relativity. Actually, conservation equation in Lyra's manifold is satisfied only on giving some special condition on displacement vector β as shown above.

3 Solutions of the Field Equations

The field equations (10)–(12) are a system of three equations with five unknown parameters R, S, p, ρ and β. Two additional constraints relating these parameters are required to obtain explicit solutions of the system. We assume that the expansion (θ) in the model is proportional to the shear (σ). This condition leads to

$$R=mS^n, \tag{18}$$

where m and n are constants and

$$\theta=\frac{2\dot{R}}{R}+\frac{\dot{S}}{S}, \tag{19}$$

$$\sigma=\frac{1}{\sqrt{3}}\left(\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right). \tag{20}$$

The motive behind assuming this condition is explained with reference to Thorne [45], the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within ≈30 per cent [46, 47]. To put more precisely, red-shift studies place the limit

$$\frac{\sigma}{H}\leq 0.3$$

on the ratio of shear σ to Hubble constant H in the neighborhood of our Galaxy today. Collins et al. [48] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition $\frac{\sigma}{\theta}$ is constant.

To solve the field equations, we follow the technique of Bali and Banerjee [49]. From (10) and (11), we obtain

$$\frac{\dot{R}\dot{S}}{RS} - \frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{S^2}{R^4} - \frac{\dot{R}^2}{R^2} = 0. \tag{21}$$

Equations (18) and (21) lead to

$$\frac{\ddot{S}}{S} + 2n\left(\frac{\dot{S}}{S}\right)^2 = \frac{1}{m(n-1)}S^{2(1-2n)}, \tag{22}$$

which reduces to

$$2\ddot{S} + 4n\frac{\dot{S}^2}{S} = \frac{2}{m(n-1)}S^{(3-4n)}. \tag{23}$$

Equation (23) takes the form as

$$\frac{d}{dS}(f^2) + \frac{4n}{S}f^2 = \frac{2}{m(n-1)}S^{(3-4n)}, \tag{24}$$

where $\dot{S} = f(S)$, $\ddot{S} = ff'$, $f' = \frac{df}{dS}$.

From (24), we obtain

$$f^2 = \dot{S}^2 = \frac{1}{2m(n-1)}S^{4(1-n)} + kS^{-4n}, \tag{25}$$

k being an integrating constant. Equation (25) leads to

$$\frac{S^{2n}dS}{\sqrt{\frac{1}{[2m(n-1)S^{-4}]} + k}} = dt. \tag{26}$$

The solution of (26) is not tenable for $n = 1$. One can choose the value of n such that above relation be integrable. We consider the following two cases.

4 First Model

When $n = \frac{1}{2}$, Equation (26) reduces to

$$\frac{\sqrt{m}SdS}{\sqrt{mk - S^4}} = dt, \tag{27}$$

which after integration leads

$$S^2 = \sqrt{mk}\sin\left\{\frac{2}{\sqrt{m}}(t+\alpha)\right\}, \tag{28}$$

where α is a constant of integration. Equation (18) for $n = \frac{1}{2}$, leads to

$$R^2 = m^{\frac{5}{4}} k^{\frac{1}{4}} \sqrt{\sin\left\{\frac{2}{\sqrt{m}}(t+\alpha)\right\}}. \tag{29}$$

Therefore, the metric (4) is reduced to

$$ds^2 = -dT^2 + m^{\frac{5}{4}} k^{\frac{1}{4}} \sqrt{\sin\left\{\frac{2T}{\sqrt{m}}\right\}}(dx^2+dz^2) + \sqrt{mk}\sin\left\{\frac{2T}{\sqrt{m}}\right\}(dy - x dz)^2, \tag{30}$$

where $t + \alpha = T$.

4.1 Some Physical and Geometric Properties of First Model

Equation (17) gives either $\beta = 0$ or $\frac{3}{2}\dot{\beta} + \frac{3}{2}\beta(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}) = 0$. Therefore

$$\frac{\dot{\beta}}{\beta} = -\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right), \tag{31}$$

which reduces to

$$\frac{\dot{\beta}}{\beta} = -\frac{2}{\sqrt{m}}\cot\left(\frac{2T}{\sqrt{m}}\right). \tag{32}$$

Integrating (32), we obtain

$$\beta = \text{Cosec}\left(\frac{2T}{\sqrt{m}}\right). \tag{33}$$

The expressions for pressure (p) and density (ρ) for the model (30) are given by

$$8\pi p = \frac{5}{4m}\cot^2\left(\frac{2T}{\sqrt{m}}\right) + \frac{3}{m} - \frac{1}{4m^2} - \frac{3}{4}\text{Cosec}^2\left(\frac{2T}{\sqrt{m}}\right), \tag{34}$$

$$8\pi\rho = \frac{1}{m}\cot^2\left(\frac{2T}{\sqrt{m}}\right) - \frac{1}{4m^2} + \frac{3}{4}\text{Cosec}^2\left(\frac{2T}{\sqrt{m}}\right). \tag{35}$$

From (35), we note that $\rho(t)$ is a decreasing function of time and $\rho > 0$ for $m \geq \frac{1}{4}$. Figure 1 also shows this behavior of energy density. This is to be taken as a representative case of physical viability of the first model.

Halford [4] has pointed out that the displacement field ϕ_i in Lyra's manifold plays the role of cosmological constant Λ in the normal general relativistic treatment. From (33), it is observed that the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford as well as with the recent observations [50–54] leading to the conclusion that $\Lambda(t)$ is a decreasing function of t.

With regard to the kinematic properties of the velocity vector v^i for the model (30), a straight forward calculation leads to the expressions for the scalar of expansion θ, Hubble's

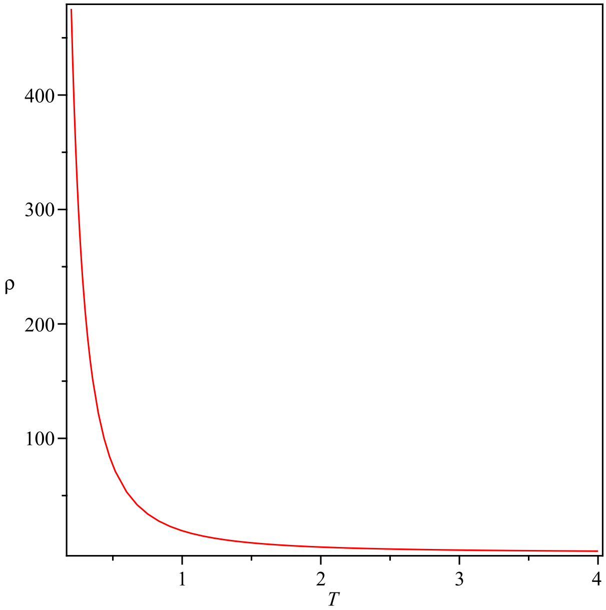

Fig. 1 Energy density ρ versus T

parameter H, shear σ, deceleration parameter q and proper volume V^3 of the fluid:

$$\theta = \frac{2}{\sqrt{m}} \cot\left(\frac{2T}{\sqrt{m}}\right), \tag{36}$$

$$H = \frac{2}{3\sqrt{m}} \cot\left(\frac{2T}{\sqrt{m}}\right), \tag{37}$$

$$\sigma = \frac{1}{2m\sqrt{3}} \left|\cot\left(\frac{2T}{\sqrt{m}}\right)\right|, \tag{38}$$

$$q = -\frac{V\ddot{V}}{\dot{V}^2} = -1 + 3\sqrt{m}\sec^2\left(\frac{2T}{\sqrt{m}}\right), \tag{39}$$

$$V^3 = \sqrt{-g} = m\sqrt{mk}\sin\left(\frac{2T}{\sqrt{m}}\right). \tag{40}$$

From (39), we observe that

$$q < 0 \quad \text{if} \quad \sec^2\left(\frac{2T}{\sqrt{m}}\right) < \frac{1}{3\sqrt{m}},$$

and

$$q > 0 \quad \text{if} \quad \sec^2\left(\frac{2T}{\sqrt{m}}\right) > \frac{1}{3\sqrt{m}}.$$

The model (30) starts with a big bang at $T = 0$ and the expansion in the model decreases as time increases. However, the expansion in the model stops when $T = \frac{\pi\sqrt{m}}{4}$. The model, in general, represents an expanding, shearing and non-rotating universe. The spatial volume increases as time increases. Since $\frac{\sigma}{\theta}$ = constant, the model does not approach isotropy.

There is a Point Type singularity in the model at $T=0$ (MacCallum [55]). For the condition $\sec^2(\frac{2T}{\sqrt{m}}) < \frac{1}{3\sqrt{m}}$, the solution gives accelerating model of the universe. It can be easily seen that when $\sec^2(\frac{2T}{\sqrt{m}}) > \frac{1}{3\sqrt{m}}$, our solution represents decelerating model of the universe. It is remarkable to mention here that the model (30) involves periodic functions and gives rise to cyclic mode of expansion.

5 Second Model

When $n=\frac{3}{2}$, Equation (26) reduces to

$$\frac{\sqrt{m}S^3 dS}{\sqrt{S^4+mk}} = dt, \tag{41}$$

which, after integration, leads to

$$S^4 = \frac{4}{m}(t+K)^2 - mk, \tag{42}$$

where K is the constant of integration. Equation (18) for $n=\frac{3}{2}$ leads to

$$R^2 = m^2 S^3 = m^2\left[\frac{4(t+K)^2}{m} - mk\right]^{\frac{3}{4}}. \tag{43}$$

Therefore, the metric (4) reduces to the form

$$ds^2 = -dT^2 + m^{\frac{5}{4}}(4T^2 - m^2k)^{\frac{3}{4}}(dx^2+dz^2) + \frac{1}{\sqrt{m}}(4T^2-m^2k)^{\frac{1}{2}}(dy - xdz)^2, \tag{44}$$

where $t+K=T$.

5.1 Some Physical and Geometric Properties of Second Model

In the second case (31) reduces to

$$\frac{\dot{\beta}}{\beta} = -\frac{16}{m}\frac{T}{[\frac{4}{m}T^2 - mk]}. \tag{45}$$

Integrating (45), we obtain

$$\beta = \frac{m^2}{(4T^2 - m^2k)^2}. \tag{46}$$

The expressions for pressure (p) and density (ρ) for the model (44) are given by

$$8\pi p = \frac{T^2+5m^2k}{(4T^2-m^2k)^2} - \frac{1}{4m^3(4T^2-m^2k)} - \frac{3m^4}{4(4T^2-m^2k)^4}, \tag{47}$$

$$8\pi\rho = \frac{21T^2}{(4T^2-m^2k)^2} - \frac{1}{4m^3(4T^2-m^2k)} + \frac{3m^4}{4(4T^2-m^2k)^4}. \tag{48}$$

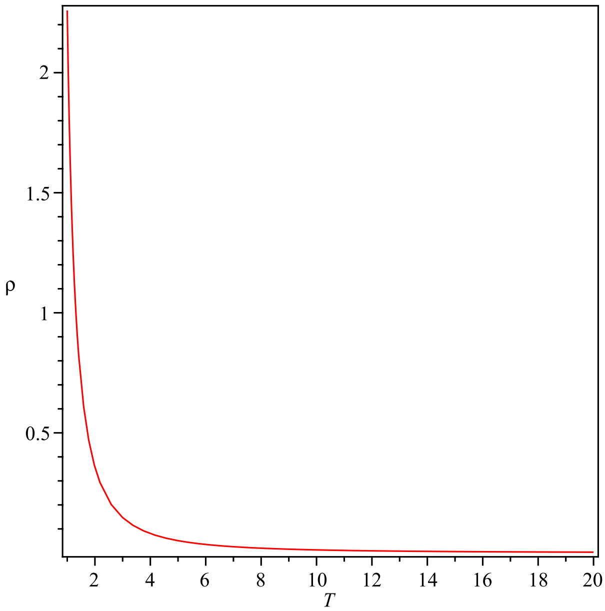

Fig. 2 Energy density ρ versus T

With regard to the kinematic properties of the velocity vector v^i for the model (44), we obtain the expressions for the scalar of expansion θ, Hubble's parameter H, shear σ, deceleration parameter q and proper volume V^3 of the fluid as

$$\theta = \frac{8T}{(4T^2 - m^2k)}, \tag{49}$$

$$H = \frac{8T}{3(4T^2 - m^2k)}, \tag{50}$$

$$\sigma = \frac{T}{\sqrt{3}(4T^2 - m^2k)}, \tag{51}$$

$$q = \frac{1}{2} + \frac{3m^2k}{8T^2}, \tag{52}$$

$$V^3 = m(4T^2 - m^2k). \tag{53}$$

The reality of the energy density depends on the values of constants m and k. From (48), we observe that the energy density is a decreasing function of time and it is always positive. Figure 2 also shows this behavior of energy density. The model (44) starts with a big bang at $T = \frac{m\sqrt{k}}{2}$ and the expansion in the model decreases as time increases. In general, the model represents an expanding, shearing and non-rotating universe. Since $\frac{\sigma}{\theta} =$ constant, the model does not approach isotropy. The spatial volume increases as time increases. There is a Point Type singularity in the model at $T = \frac{m\sqrt{k}}{2}$ (MacCallum [55]). From (46), it is observed that the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford as well as with the recent observations [50–54] leading to the conclusion that $\Lambda(t)$ is a decreasing function of t.

6 Entropy in the Universe

In this section we discuss entropy in our derived universe. In thermodynamics, the expression for entropy is given by

$$dS = d(\rho V^3) + p(dV^3), \tag{54}$$

where $V^3 = R^2 S$ is proper volume in our case. To solve the entropy problem of standard model, it is necessary to treat $dS > 0$ for at least a part of evolution of the universe. Hence (54) reduces to

$$TdS = \dot{\rho} + (\rho + p)\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) > 0. \tag{55}$$

The conservation equation $T^{j}_{1;j} = 0$ for (4) leads to

$$\dot{\rho} + (\rho + p)\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) + \frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = 0. \tag{56}$$

Therefore, (55) and (56) lead to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) < 0, \tag{57}$$

which gives $\beta < 0$. Since for a physical model of the universe $\beta(t) > 0$, thus we observe that the entropy affects the displacement vector because for entropy $dS > 0$ leads to $\beta(t) < 0$. It is remarkable to mention here that we obtain the displacement vector $\beta(t)$ in our derived universe as decreasing function of time and always positive but entropy affects the displacement vector.

7 Conclusions

We have obtained a new class of LRS Bianchi type II cosmological models in presence of perfect fluid distribution of matter within the framework of normal gauge in Lyra's manifold. We have presented an alternative and straightforward approach to solve the Einstein's typical, non-linear field equations by considering the expansion in the model is proportional to the shear as Collins et al. [48] have showed that the normal congruence to the homogeneous expansion satisfies that the condition $\frac{\sigma}{\theta}$ = constant. Both models (30) and (44) start with a big bang singularity. We have obtained a Point Type singularity in both models. In general, the models represent an expanding, shearing and non-rotating universe.

It is observed that the displacement vector $\beta(t)$ matches with the nature of the cosmological constant Λ which has been supported by the work of several authors as discussed in the physical behavior of the models in Sects. 4 and 5. In recent time Λ-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a Λ-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [56] and [57]). Assuming that Λ owes its origin to vacuum interaction, as suggested particularly by Sakharov [58], it follows that it would, in general, be a function of space and time coordinates, rather than a strict constant. In a homogeneous universe Λ

will be at most time dependent [59]. In recent past there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [60–62]. Therefore the study of cosmological models in Lyra's manifold may be relevant for inflationary models. There seems a good possibility of Lyra's manifold to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. However, the importance of Lyra's manifold for astrophysical bodies is still an open question. In fact, it needs a fair trial for experiment.

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